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Generating Random Ray Distributions

2023 04 04, rev. 2023 09 26

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Background

The motivation for this note was to formulate a method for generating a circular 2-dimensional random ray distribution (field points in star apace) for a seeing simulator, among others.

This removes the need for intensity weighting and means that throughput calculations are simply a matter of counting transmitted rays.

Random rays with a uniform intensity distribution can also be generated using the same method.

Theory

Generating a uniform random ray distribution over a rectangle is trivial. The differential area is given by:

$$dA = dx \cdot dy$$

And uniform random points can be drawn from the respective variable domains. This is often described as "point-picking".

We seek to reduce any point picking problem to a uniform distribution in a rectangular domain, thereby rendering it trivial.

Circular Area

For a circular area, and switching to polar coordinates with conventional notation, the differential area (sometimes called the "polar rectangle") becomes:

$$dA = \rho \cdot d\rho \cdot d\theta$$

We can transform variables to a rectangular domain [where $\boldsymbol{u}$ is the uniform random variable] thus:

$$dA = du \cdot d\theta, \quad where$$

$$du = \rho \cdot d\rho$$

whence

$$u = \frac{r^2}{2}$$

after integrating from 0 to r. For convenience, we can rescale to the unit circle (as per ref [1]) and draw random points from the domains of $(u,\ \theta)$, to find:

$$r = \sqrt{u}\ \ , (u\text{: } [0,1] \rightarrow r\text{: } [0,1])\ , (\theta\text{: } [0,2\pi])$$

And rescale and convert to rectangular coordinates in the normal way.

Example: Seeing Simulation

The normalized 2D Moffat function for an atmospheric seeing profile looks like:

$$I(\rho) = \frac{2(\beta - 1)}{2\pi\alpha^2}\left(1 + \frac{\rho^2}{\alpha^2}\right)^{-\beta}$$

$$where\ \rho\ is\ the\ radial\ parameter, and:$$

$$\alpha = \frac{fwhm}{2\sqrt{2^{\frac{1}{\beta}} - 1}}$$

This is a modified Lorentzian function, sometimes described as a “softened Gaussian”. The radial parameter is usually measured in arcseconds in star space.

Normally:

$$\beta = 4$$

The differential area is now weighted by the intensity:

$$dA = \rho \cdot \mathrm{I}(\rho) \cdot d\rho \cdot d\theta$$

Transform variables, as before, to a rectangular domain, and ignoring the already rectangular coordinate, Θ (which cancels the 2π in the denominator) :

$$du = \rho \cdot \mathrm{I}(\rho) \cdot d\rho$$

Whence:

$$u = \frac{2(\beta - 1)}{\alpha^2}\left(1 + \frac{\rho^2}{\alpha^2}\right)^{-\beta+1} \frac{\alpha^2}{2(-\beta + 1)} = -\left(1 + \frac{\rho^2}{\alpha^2}\right)^{-\beta+1}$$

And so:

$$u = \left[1 - \left(1 + \frac{r^2}{\alpha^2}\right)^{-\beta+1}\right]$$

after integrating from 0 to r. In other words, at what value of r is proportion u of the seeing energy contained?

Thus:

$$u = \left[1 - \left(1 + \frac{r^2}{\alpha^2}\right)^{-\beta+1}\right], \quad u: [0,1] \rightarrow r: [0, \infty]$$

And so, the uniform random variable, u , can be mapped to the radial coordinate, r , weighted by an intensity distribution. Solving for r :

$$r = \alpha\sqrt{(1 - u)^{\frac{1}{1-\beta}} - 1}\ ,\ \ 0 \leq u < 1$$

We just have to be careful and catch the exception should the random variable ever evaluate to exactly unity.

The conversion to rectangular coordinates (field points in star space) proceeds as before.

We can test this with around 100,000 field points over a seeing disc. Field points can be counted, binned, and compared with the theoretical distribution.

Over about 4.5x the seeing FWHM, and using 21 x 21 bins, the convergence is uncanny.

Example: Gaussian Beam

The 2D intensity profile of a circular Gaussian beam at the beam waist is given by:

$$I(\omega) = \frac{2}{\pi\omega_0^2} e^{-2\left(\frac{\omega}{\omega_0}\right)^2}$$

$$where\ \omega\ is\ the\ radial\ parameter, and:$$

$$\omega_0 = beam\ waist\ radius\ at\ e^{-2}\ of\ the\ central\ intensity$$

We would use this profile, for example, to model rays from a sample illuminated by a focussed laser beam.

The differential area is now weighted by the intensity:

$$dA = \omega \cdot \mathrm{I}(\omega) \cdot d\omega \cdot d\theta$$

Transform variables, as before, to a rectangular domain:

$$du = \omega \cdot \mathrm{I}(\omega) \cdot d\omega$$

And after losing the π in the denominator and ignoring the constant of integration:

$$u = -e^{-2\left(\frac{\omega}{\omega_0}\right)^2}$$

And so:

$$u = \left[1 - e^{-2\left(\frac{r}{\omega_0}\right)^2}\right], \quad u: [0,1] \to r: [0, \infty]$$

after integrating from 0 to r as before. After rearranging:

$$r = \omega_0 \sqrt{-\frac{1}{2} ln(1-u)}, \;\; 0 \le u < 1$$

As before, we have to catch the exception should the random variable ever evaluate to exactly unity.

More often than not the illuminated area is an ellipse with a double-waisted profile looking like:

$$I(x,y) = \frac{2}{\pi\omega_x\omega_y} e^{-2\left[\left(\frac{x}{\omega_x}\right)^2 + \left(\frac{y}{\omega_y}\right)^2\right]}$$

Integrating this to find the relationship between random variables ***u*** and ***v***, and ***x*** and ***y***, is not quite so straightforward and of course means inverting ***erf(x)*** and ***erf(y)***, which can get a little fiddly.

The easiest way around this is to derive the random points for the circular distribution and then scale the ***x*** or ***y***-coordinate (depending on the direction of the ellipticity) after the conversion to rectangular coordinates.

Similar testing shows nice congruence with both the elliptical and circular intensity profiles.

Hemispherical and Spherical Surfaces

In spherical coordinates and using conventional (mathematical) notation, on a unit sphere, the differential area is:

$$dA = \sin\phi \cdot d\phi \cdot d\theta$$

Transform to a rectangular domain, ignoring Θ as before:

$$du = \sin\phi \cdot d\phi$$

Whence, for a hemisphere:

$$u = 1 - \cos\phi \ , u: [0,1] \to \phi: [0, \pi/2]$$
$$\cos\phi = 1 - u$$
$$\sin\phi = \sqrt{u(2-u)}$$

And, for a sphere:

$$u = \frac{1 - \cos\phi}{2} \ , u: [0,1] \to \phi: [0, \pi]$$
$$\cos\phi = 1 - 2u$$
$$\sin\phi = 2\sqrt{u(1-u)}$$

Note that the constants of integration and the scale factors have been chosen to align the domains in a logical way. And of course, for a sphere of arbitrary radius:

$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

On a unit sphere the coordinates would be simply direction cosines.

Example: LED Ray Distribution

Hemispherical point picking is especially useful for generating random rays from a point source or an LED using the published ray intensity weightings as a function of the polar angle.

We can weight the differential area as before, as a function of the polar angle:

$$du = \sin\phi \cdot \mathrm{I}(\phi) \cdot d\phi$$

$I(\phi)$ can be approximated by several types of basis functions but the above resulting product should preferably be integrable. One would then use Newton's method to solve for ϕ starting at a solve for the first term, if the approximation is a series.

These techniques for an LED source are yet to be tested.

Disclaimer

I'm not making any claims for originality and pure mathematical correctness, but if something works, I'll use it, and hope to help others.

As always, if you have a problem or constructive criticism, please drop me an email.

References

(1) "Point Picking and Distributing on the Disc and Sphere", Mary K Arthur, U.S. Army Research Laboratory, 2015, ARL-TR-7333, https://apps.dtic.mil/sti/pdfs/ADA626479.pdf
[Equations (41) contain a series of typos, the set in the text above has been corrected]